

# Ferromagnetic neutron stars: axial anomaly, dense neutron matter, and pionic wall

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We show that a chiral nonlinear sigma model coupled to degenerate neutrons exhibits a ferromagnetic phase at high density. The magnetization is due to the axial anomaly acting on the parallel layers of neutral pion domain walls spontaneously formed at high density. The emergent magnetic field would reach the QCD scale  $\sim 10^{19}$  [G], which suggests that the quantum anomaly can be a microscopic origin of the magnetars (highly magnetized neutron stars).

**Magnetars and high density neutron matter.** — The phase diagram of QCD is a mystery to be uncovered. Even though the problem is theoretically well-posed with QCD Lagrangian, so far the difficulty remains yet to reveal a specific region of the phase diagram, the region at low temperature and high baryon number density  $\rho$  [1].

One of the promising systems where such a high density region is realized in nature is the deep interior of compact stars, such as the neutron stars. Observations of various properties of these stars should give us crucial constraint on high density matter. In particular, the magnetars, which are considered to be neutron stars with very strong magnetic field  $\sim 10^{15}$  [G] at their surface, are of particular interests [2, 3]. The mechanism for generating such a strong magnetic field is so far not established: Among various proposals including the dynamo formation model and the fossil-field model [2], nuclear ferromagnetism associated with the solidification of the neutron star core, which was originally proposed right after the discovery of the pulsar [4], is an interesting possibility to generate large intrinsic magnetic field. However, modern quantum many-body calculations on the neutron matter and asymmetric nuclear matter with realistic nuclear force have shown that these systems stay in the liquid phase at high density without having spontaneous ferromagnetic transition [5]. Another interesting possibility is the ferromagnetism of the quark liquid in the central core of neutron star [6]; spin-polarized quark phase, similar to that in the low density electron gas, may be realized in a certain window of baryon density due to the Fock term of the gluon exchange between quarks.

In this letter, we propose a novel mechanism which leads to a *spontaneous* magnetization of the neutron matter, based on the non-linear chiral Lagrangian of pions coupled to degenerate neutrons. Two basic ingredients are (i) the neutron spin-density induced on a pion domain wall in dense matter [7] and (ii) the baryon number induced on the pion domain wall by an external magnetic field through axial anomaly [8]. We show that layers of  $\pi^0$  domain walls are spontaneously generated by a small seed of an external magnetic field. Then, intrinsic magnetic field is induced from the layers which have net

magnetization. If this mechanism takes place inside the core of the neutron stars above certain threshold density, they acquire large magnetic field and become magnetars.

**Chiral Lagrangian and pion domain walls.** — We use the chiral Lagrangian for low energy pions and nucleons with the Weinberg parametrization [9]:

$$\mathcal{L} = \bar{N} [i\gamma^\mu (\partial_\mu + i\boldsymbol{\tau} \cdot \mathbf{V}_\mu + i\gamma_5 \boldsymbol{\tau} \cdot \mathbf{A}_\mu) - m_N] N + \frac{1}{2} |D_\mu \phi|^2 - \frac{1}{2} (m_\pi^2 + \sigma_{\pi N} \bar{N} N) \frac{\phi^2}{1 + \phi^2/4f_\pi^2}. \quad (1)$$

Here  $N$  is the nucleon field (isospin  $SU(2)$  doublet),  $\phi$  is the pion (triplet),  $f_\pi$  is the pion decay constant,  $g_A$  is the axial charge of the nucleon, and  $m_\pi$  is the pion mass. Also,  $\mathbf{V}_\mu \equiv \frac{1}{4f_\pi^2} (1 + \phi^2/4f_\pi^2)^{-1} (\phi \times \partial_\mu \phi)$ ,  $\mathbf{A}_\mu \equiv \frac{g_A}{2f_\pi} D_\mu \phi$  with  $D_\mu \phi \equiv (1 + \phi^2/4f_\pi^2)^{-1} \partial_\mu \phi$ , respectively.

In [7], neutrons are integrated with neutron chemical potential, and  $\pi^0$  domain wall solutions were studied. Here we generalize it to include the charged pion and obtain the in-medium chiral Lagrangian up to  $O(p^2)$ ;

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{\alpha}{2} |D_0 \phi_3|^2 - \frac{\beta}{2} |D_i \phi_3|^2 + \frac{\tilde{\alpha}}{2} |D_0 \phi_+|^2 - \frac{\tilde{\beta}}{2} |D_i \phi_+|^2 \\ & - \gamma_0 \frac{m_\pi^2}{2} \phi^2 / (1 + \phi^2/4f_\pi^2) \\ & + \gamma_1 (\phi_+ (D_0 \phi_+)^* - (\phi_+)^* D_0 \phi_+) \\ & + \gamma_2 (\phi_+ (D_0 \phi_+)^* - (\phi_+)^* D_0 \phi_+)^2 \\ & + \gamma_3 |\phi_+ D_0 \phi_3 - \phi_3 D_0 \phi_+|^2 + \gamma_4 |\phi_+ D_i \phi_3 - \phi_3 D_i \phi_+|^2 \end{aligned} \quad (2)$$

with  $i = 1, 2, 3$ . We have defined the charged pion  $\phi_+ \equiv \phi_1 + i\phi_2$ , and  $D_\mu \phi_+ \equiv (\partial_\mu \phi_+ + i\delta_{\mu 0} \mu_I \phi_+) / (1 + |\phi|^2/4f_\pi^2)$ , where  $\mu_I$  is the isospin chemical potential defined as the difference  $\mu_I \equiv \mu_p - \mu_n$ . The correction due to the background neutrons is in the coefficients  $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$  (and  $\gamma_{1,2,3,4}$ ), which are equal to the unity (zero) in the absence of the nucleons. They are given by one-loop calculations (see [7] for  $\alpha$  and  $\beta$ )

$$\begin{aligned} \beta & \equiv 1 - \frac{g_A^2 m_N^2}{4\pi^2 f_\pi^2} \log(x + \sqrt{x^2 - 1}), \\ \alpha & \equiv 1 + \frac{g_A^2 m_N^2}{4\pi^2 f_\pi^2} (x\sqrt{x^2 - 1} - \log(x + \sqrt{x^2 - 1})), \end{aligned}$$

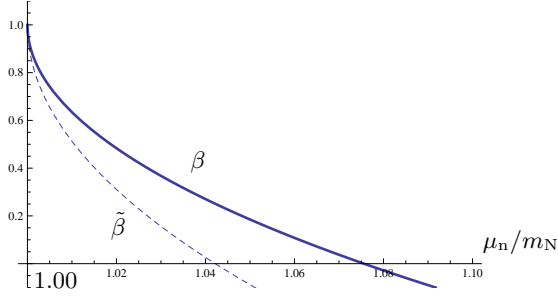


FIG. 1: Plots of  $\beta$  (rigid lines) and  $\tilde{\beta}$  (dashed lines) as a function of  $\mu_n$ .  $\beta$  decreases monotonically as the density grows. Here we used  $m_N = 940$  [MeV],  $f_\pi = 93$ [MeV] and  $g_A = 1$  (with which  $\beta$  vanishes at  $\mu_n \sim 1.08m_N$  corresponding to  $\rho_n \sim 0.23$  [fm $^{-3}$ ]). In general in dense matter  $m_N$  and  $f_\pi$  may vary, so the plot should be understood only qualitatively.  $\alpha$  and  $\tilde{\alpha}$  do not vary much in density.

$$\begin{aligned}\tilde{\beta} &\equiv 1 - \frac{g_A^2 m_N^2}{2\pi^2 f_\pi^2} \int_1^x \frac{(s^2-1)^{1/2}(2s^2+4+3(1-x)s)}{3(1-x+2s)(x-1)} ds, \\ \tilde{\alpha} &\equiv 1 + \frac{g_A^2 m_N^2}{2\pi^2 f_\pi^2} \int_1^x \frac{(s^2-1)^{1/2}(2s^2-2+(1-x)s)}{(1-x+2s)(x-1)} ds, \\ \gamma_0 &\equiv 1 + \frac{\sigma_{\pi N}}{m_\pi^2} \frac{m_N^3}{2\pi^2} (x\sqrt{x^2-1} - \log(x+\sqrt{x^2-1})), \\ \gamma_1 &\equiv \frac{im_N^2}{24\pi^2 f_\pi^2} (x^2-1)^{3/2}, \quad \gamma_2 \equiv \frac{-m_N^2}{128\pi^2 f_\pi^4} x\sqrt{x^2-1}, \\ \gamma_3 &\equiv \frac{m_N^2}{16\pi^2 f_\pi^4} \int_1^x \frac{(s^2-1)^{1/2}(2s^2+(1-x)s)}{(1-x+2s)(x-1)} ds, \\ \gamma_4 &\equiv \frac{m_N^2}{16\pi^2 f_\pi^4} \int_1^x \frac{(s^2-1)^{1/2}(2s^2-2+3(1-x)s)}{3(1-x+2s)(x-1)} ds.\end{aligned}$$

Here  $x \equiv \mu_n/m_N$ , and we consider the pure neutron matter for simplicity, so that  $\mu_p = m_N$ . The neutron density is related to the Fermi momentum as  $\rho_n = k_F^3/(3\pi^2)$  with  $k_F \equiv \sqrt{\mu_n^2 - m_N^2}$ . The  $\sigma_{\pi N}$  correction is taken care of at its leading order. An important feature of the corrections is that  $\beta$  is a monotonically decreasing function of the neutron density, and vanishes at a certain density (Fig. 1).

A classical solution of Eq.(2) is a domain wall of the in-medium neutral pion [7],

$$\phi_3 = \frac{2f_\pi}{\sinh[m_\pi x^3/\sqrt{\beta/\gamma_0}]}, \quad \phi_1 = \phi_2 = 0. \quad (3)$$

Note that this  $\pi^0$  domain wall interpolates the vacua  $\theta = 0$  and  $\theta = 2\pi$ , where  $\tan \frac{\theta}{2} \equiv |\phi|/2f_\pi$ . Interestingly, the domain wall can reduce its weight in the neutron matter:  $\gamma_0$  stays positive at all  $x$ , while  $\beta$  approaches zero as shown in Fig.1, so that the tension  $\mathcal{E}/S$  of the domain wall is significantly reduced for  $\beta \rightarrow 0$ ,

$$\mathcal{E}/S = 8\sqrt{\beta\gamma_0}f_\pi^2m_\pi, \quad (4)$$

where  $S$  is the domain wall area. We will show that the parallel layers of the domain walls would populate at high

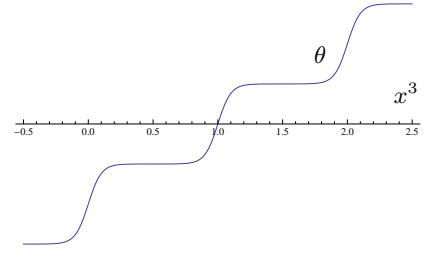


FIG. 2: The multi domain wall solution written by  $\theta(x^3)$ . Each wall interpolates adjacent vacua,  $\theta = 0, 2\pi, 4\pi, \dots$ . For the exact solution, see for example [10].

baryon density due to the reduction of the tension and the accumulation of baryon number on the domain wall.

**Emergent magnetic field on pionic walls from axial anomaly.** — Let us first discuss how the spontaneous magnetization occurs in high density neutron matter. There are three steps for this to happen: (i) The in-medium domain wall becomes light, and at the same time it acquires finite baryon density due to axial anomaly under an external magnetic field. Then the system with a domain wall becomes energetically favorable than the uniform neutron matter above a certain density. (ii) The domain wall is magnetized due to the spin alignment of surrounding neutrons, so that it creates spontaneous magnetization which enlarges the original magnetic field. (iii) The enhanced magnetic field creates more domain walls. The cycle (i)  $\rightarrow$  (ii)  $\rightarrow$  (iii)  $\rightarrow$  (i) is repeated and leads to a stable configuration with many parallel layers of thin domain walls with a high magnetic field. All the spins are aligned, so the system is ferromagnetic.

We now explain the mechanism of inducing the baryon charge following Ref.[8]. In a background constant magnetic field, the QCD axial anomaly term in the chiral Lagrangian reads

$$\mathcal{L}_{\text{WZW}} = \frac{ie}{16\pi^2} A_0^{(B)} B_3 \text{tr} [\tau_3 (U \partial_3 U^\dagger + \partial_3 U^\dagger U)], \quad (5)$$

where  $A_0^{(B)}$  is the temporal component of a gauge potential for the baryon number symmetry,  $B_3$  is the background magnetic field along  $x^3$ , and  $U \equiv \cos \theta + i\tau \cdot \hat{\phi} \sin \theta$  with  $\hat{\phi} = \phi/|\phi|$ . Since the pion-dependent part can be evaluated (for  $\phi_1 = \phi_2 = 0$ ) as

$$\text{tr} [\tau_3 (U \partial_3 U^\dagger + \partial_3 U^\dagger U)] = -4i \frac{D_3 \phi_3}{f_\pi} = -4i \partial_3 \theta, \quad (6)$$

we immediately see that the domain wall, which interpolates  $\theta = 0$  and  $\theta = 2\pi$ , can obtain a baryon charge per a unit area [8],

$$N_B/S = eB_3/2\pi. \quad (7)$$

If the domain wall is not parallel to the magnetic field, the induced baryon charge is reduced to  $B_i \hat{n}_i$  where  $\hat{n}$  is

the unit vector perpendicular to the domain wall. Note that the formula Eq.(7) is valid even for  $\beta \neq 1$ .

Combining Eq.(7) with Eq.(4), the domain wall energy per a unit baryon charge is

$$\mathcal{E}/N_B = \frac{16\pi f_\pi^2 m_\pi}{eB_3} \sqrt{\beta\gamma_0}. \quad (8)$$

The system with domain wall is more favorable than the uniform neutron matter when the domain wall energy per baryon becomes smaller than the neutron chemical potential [8], i.e.  $B_3 > \sqrt{\beta\gamma_0} \times (16\pi f_\pi^2 m_\pi / \mu_n) \sim \sqrt{\beta\gamma_0} \times 10^{19} [\text{G}]$ . Here, the factor  $\beta$ , which was not taken into account in [8], is important. Since  $\beta(\rho)$  is a monotonically decreasing function of  $\rho_n$ , an adiabatic increase of the density  $\rho_n$  inevitably hits the critical value of  $\beta$  at which the domain wall is created.

Once the pionic wall is formed, neutron spins on the wall align in the direction perpendicular to the domain wall [7]. The spin density of the neutrons is a spatial part of the axial current  $j_i^{(A)} = \langle \bar{\psi}_n \gamma_i \gamma_5 \psi_n \rangle$ . It was evaluated in [7], in the same approximation, as

$$s_3/S = 2\pi(\beta - 1)f_\pi^2, \quad (9)$$

which is the third component of the neutron spin density per a unit area of the domain wall.

We note here that neutrons have a magnetic moment  $\mu = ges_3/2m_N$  where  $g \sim -3.8$  is the neutron  $g$ -factor. So the total magnetic moment (which is the magnetization  $M$ ) per a unit volume at  $\beta \sim 0$  is

$$M = \frac{\pi|g|ef_\pi^2}{m_N} \frac{1}{d} \quad (10)$$

where  $d$  is the separation between adjacent domain walls. Therefore *the domain wall phase is ferromagnetic*.

The magnetization  $M$  is larger for smaller separation  $d$  among the domain walls. This  $d$  is intimately related to the induced baryon density due to the domain wall, and in fact this is a driving force for developing strong magnetic field. From Eq.(7), we know that the averaged baryon number density induced by the domain walls is

$$\rho_{\text{dw}} = \frac{eB_3}{2\pi} \frac{1}{d}. \quad (11)$$

The total baryon number density  $\rho$  is given as a sum of  $\rho_{\text{dw}}$  and the remaining neutron density  $\rho_n$ . (This  $\rho_n$  cannot vanish, since the  $\beta$  correction needs background neutrons.) Combining Eq.(11) with Eq.(10), we obtain

$$M = \frac{2\pi^2|g|f_\pi^2\rho_{\text{dw}}}{m_N B_3}. \quad (12)$$

Once the background  $B_3$  becomes larger, the domain wall energy cost (8) becomes smaller. So the domain wall is created easier, and the domain wall separation  $d$  becomes smaller. Then from Eq.(10), the magnetization  $M$

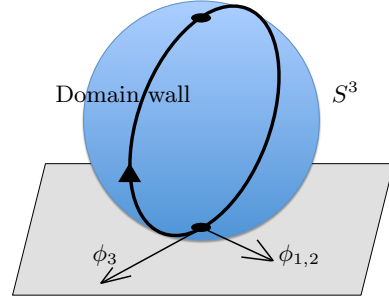


FIG. 3: The topological path of the neutral pion domain wall. The sphere represents the target space  $S^3$  of the sigma model. On the  $S^3$ , the thick line with an arrow represents the domain wall solution (3). It rounds the sphere, but topologically trivial. The plane beneath the sphere is the parameterization space of  $\phi_3$  and  $\phi_{1,2}$ . There is a one-to-one correspondence between a point on the sphere and a point on the plane.

increases and helps the  $B_3$  to increase. So, this system has a self-enhancement mechanism of the magnetic field. The equilibrium can be reached at  $B_3 = M$  which is our critical induced magnetic field,

$$B_3 = \sqrt{2\pi^2|g|f_\pi^2\rho_{\text{dw}}/m_N} \sim 3 \times 10^{19} [\text{G}]. \quad (13)$$

supposing a typical value for  $\rho \sim \rho_{\text{dw}} \sim \rho_n$ . The value of the magnetic field is quite large ( $\sqrt{B_3} \sim 10^2 [\text{MeV}]$ ) and close to the QCD deconfinement scale around which our approximation breaks down[15]. The magnetization is expected to stop increasing somewhere before reaching this value.

In addition to the neutron spin alignment, the WZW term (5) itself may provide a magnetization [13] of the same sign. Details will be in our forthcoming paper [14].

Finally we briefly comment on the stability of the domain walls. Our domain wall is topologically trivial, because the target space of the non-linear sigma model is  $SU(2) \simeq S^3$  on which the two vacua  $\theta = 0, 2\pi$  are the same point (south pole): See Fig. 2 and Fig. 3. Therefore, the domain walls can be created spontaneously, but on the other hand, they can decay through the fluctuations along  $\phi_{1,2}$  directions. Indeed, it was shown that the domain wall is stable only for  $B_3 > 10^{19} [\text{G}]$  at  $\beta = \tilde{\beta} = 1$  by analyzing its local and global stability [8]. As we have shown after Eq.(13), the above condition is replaced by  $B_3 > \sqrt{\beta\gamma_0} \times 10^{19} [\text{G}]$ . This implies that the global stability is guaranteed for smaller  $B_3$  when  $\beta \rightarrow 0$ . The analysis of the local stability is more involved especially where  $\beta > 0$  and  $\tilde{\beta} < 0$  (see Fig.1): In this case, the charged pion condensation  $\phi_+ = a \exp(-i\mu_\pi t + i\vec{k} \cdot \vec{x})$  should be considered together with the  $\pi^0$  domain wall. We leave the analysis to our future work.[16]

**Implication to magnetars.** — A schematic picture of the core region of the neutron star is shown in Fig. 4. Only at the core region, because of the high density (or

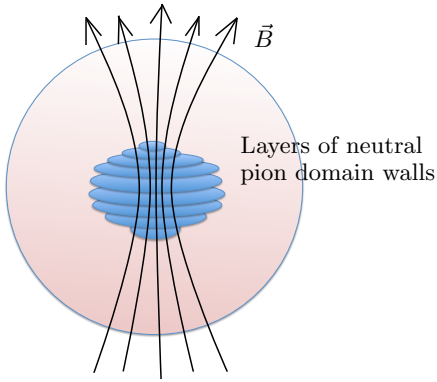


FIG. 4: A schematic figure of the neutron star with domain wall layers at the core. Scales should not be taken seriously.

rather to say the large value of the neutron chemical potential), the domain walls are present. At the boundary of the domain walls, neutrons drip from the wall boundary so that the total baryon number is conserved [11]. As the domain wall layers, the ferromagnetic region, are present only at the core of the neutron star, at the surface of the neutron star the magnetic field does not reach the value in Eq.(13), but it would be enough strong to explain the magnetars.

The magnetic field at the surface of the neutron star is smaller than that of the domain wall core, as  $B_{\text{surface}} = B_{\text{core}}(R_{\text{core}}/R_{\text{NS}})^3$  at the North pole.  $R_{\text{core}}$  is the radius of the domain wall core (assumed to be spherical and to have a homogeneous ferromagnetism inside), and  $R_{\text{NS}}$  is the radius of the neutron star. The average of the magnetic field magnitude on the neutron star surface is  $B_{\text{average}} = 0.69B_{\text{surface}}$ . The core-radius dependence of the surface magnetic field is shown in Fig. 5. If the core is sufficiently small such as  $R_{\text{core}}/R_{\text{NS}} \sim 1/10$ , the surface magnetic field may reduce to  $B_{\text{surface}} \sim \mathcal{O}(10^{16})$  [G].

The mechanism suggests that there are two kinds of neutron stars: one which reaches the critical density and has the domain wall layer structure, and the other which does not have it. The former would have a strong magnetic field but the latter would not have it. It is interesting that the recent data [3] of the magnitude of the magnetic field on the surface of the neutron stars show two categories, magnetars and the others.

It is important to construct more realistic models with nuclear forces, as our model uses free neutrons. For example, the neutron superfluidity can co-exist with the domain wall, since at higher densities the spin-aligned neutron pairing  $^3P_2$  is known to be favored. Furthermore, structure of the solitonic core of neutron stars would influence the equation of state, and may be sensitive to the mass-radius map of the neutron stars. If the orientation of the solitonic core is different from the rotation axis of the neutron star, it would be a source of gravitational waves. All details need to be explored, to match

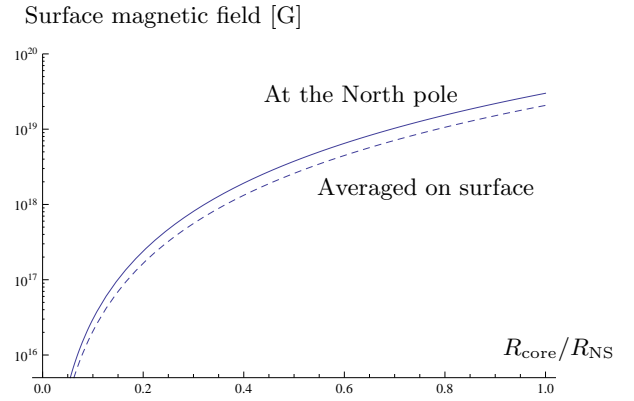


FIG. 5: The magnetic field at the neutron star surface.  $R_{\text{core}}$  is the radius of the domain wall layer core (assumed to be spherical), and  $R_{\text{surface}}$  is the radius of the neutron star. At the core, the critical magnetic field (13) is assumed. For the  $1.41M_{\odot}$  neutron star with the standard APR equation of state [12],  $R_{\text{core}}/R_{\text{NS}} = 0.1$  corresponds to the critical density of the pionic wall formation  $3.2\rho_0$ .

the observations of the neutron stars. We hope that our mechanism may survive various corrections, and explain observations.

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- [15] We can estimate from these equations that the equilibrium magnetic field corresponds to  $d \sim 0.1[\text{fm}]$ , at which our low-energy approximation is not applicable.
- [16] With the condensation, the effective action for  $\phi_3$  in Eq.(2) has a spatial kinetic term with a replacement  $\beta \rightarrow (\beta - 2\gamma_4 a^2)/(1 + a^2/(4f_\pi^2))^2$ . Here  $\gamma_4$  is negative ( $\simeq -0.006$ ) at the critical  $\mu_n$  giving  $\beta = 0$ . So, the charged pion condensation  $a \neq 0$  pushes the nearly-vanishing  $\beta$  back to a positive nonzero, which increases the domain wall tension. Hence the charged pion tends to vanish inside the wall, and the wall is expected to be stable against the charged pion fluctuations.